Estimating Equilibration Times and Heating/Cooling Rates in Heat Treatment of Workpieces with Arbitrary Geometry

M. Gao, C.N. Reid, M. Jahedi, and Y. Li

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Estimation of both equilibration times and heating/cooling rates is very important in heat treatment practices. An improved, practical virtual sphere method recently developed by two of the authors, Gao and Reid, based on the theory of heat conduction in solids, is further examined and evaluated in this article. The method is capable of estimating equilibration times for any workpiece with arbitrary geometry. The requirement of input data for this method is kept to a minimum. The method is simple and cost effective to use and offers much better accuracy than those methods based on rules of thumb. The engineering approach of applying the virtual sphere method for determining both equilibration times and heating/cooling rates at the center is provided in this article. It can be easily implemented in a computer program, and hand calculations are also possible. The test cases presented in this article for evaluating the method include a long cylinder, an infinite plate, and a short cylinder with exact analytical solutions and a H13 die with experimentally measured temperature history for a heat treatment cycle. The method is recommended for engineering applications.

Keywords equilibration time, heat conduction, heating rate, heat treatment

1. Introduction

In heat treatment, a workpiece is heated in a furnace for a given time at a given temperature. The equilibration time is defined as the time required for the workpiece to reach a uniform temperature distribution, equal to the air temperature in the furnace. Equilibration times and heating/cooling rates are two important operating parameters in the heat treatment of metals.

In some cases, it is possible to measure the surface and core temperatures to determine the equilibration time. However, this is not always possible for a variety of reasons. Instead, it is common to use some simple rules of thumb to estimate the time required for a given section size by assuming that there is a linear relationship between the equilibration time and the thickness. In theory, the relationship between the equilibration time and thickness is not always linear. In addition, the thickness is also difficult to define for a complex geometry. The rules of thumb are simple to use, but each applies only to a specific material and is inaccurate.

Most heat transfer situations in heat treatment are transient heat conduction problems with convective and radiative boundary conditions at the surfaces of the solid. In the theory of heat conduction in solids, there are at least three methods that are readily available for estimating the equilibration time. These are summarized briefly as follows.

Numerical methods: These methods solve the basic governing equation of heat conduction. With available commercial software, the methods apply to almost any geometries and nonlinear heat conduction problems. If properly used, the methods are accurate and powerful. However, these methods are expensive and not readily available for many practical engineers.

Analytical solutions: For many workpieces of regular geometry, analytical or approximate analytical solutions can be obtained for linear problems, for example, infinite flat plates, cylinders, and spheres. Some multidimensional geometries can be treated by the method of separation of variables, and the solutions of two- and three-dimensional transient heat conduction problems can be obtained by a simple product superposition of the solutions of certain one-dimensional problems. Generally, these analytical solutions are also very complex and can only be obtained by some numerical methods.

Heisler charts: Heisler (for example, Ref 1) first presented in 1947 the analytical solutions for the three most important geometries in a graphical form. The three cases are (a) plates with small thicknesses in relation to the other dimensions, (b) cylinders where the diameter is small compared to the height, and (c) spheres. Again, the solutions of two- and three-dimensional transient heat conduction problems can be obtained by a simple product superposition of the solutions of certain one-dimensional problems. The methods are simple, but can be complex for three-dimensional problems. The methods are generally not applicable to arbitrary geometries.

It should be mentioned that the convective heat transfer between the workpiece and the surrounding air is a very difficult problem to deal with for all three methods. Recently, Gao and Reid[2] developed a simple and practical virtual sphere method for estimating the equilibration times for heat treatment based on the theory of heat conduction in solids. The two most important components in this new approach are the virtual sphere concept of representing any geometry and two new simple approximate solutions for the center temperature of a solid sphere. The method was evaluated against the exact solutions of some regular geometries by Gao and Reid. However, no experimental evaluation of the method was undertaken.

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While a previous article by Gao and Reid[2] concentrated on the theoretical aspects of the method, this article emphasizes the practical aspects. The purposes of this article are basically twofold: (1) examine and summarize the virtual sphere method for determining equilibration times and heating/cooling rates from an engineer's point of view; and (2) evaluate the virtual sphere method against both experimentally measured temperature profiles in a heat treatment cycle and exact solutions for some nonspherical geometries.

The practical methods for predicting equilibration times and center temperatures will be summarized first. Advantages and disadvantages of the methods will also be discussed, followed by the physical background behind the methods. The test cases will be presented next, and, finally, some conclusions will be drawn.

2. Summary of the New Methods

2.1 Definition of the Heat Transfer Problem

A workpiece with arbitrary geometry has a volume of V (m^3) and a total heat transfer surface of area A (m^2). The density, specific heat, and thermal conductivity of the workpiece are ρ (kg/m^3), c (J/kg · °C), and k (W/m · °C), respectively. The overall heat transfer coefficient to the surface is h ($W/m^2 \cdot °C$). The initial temperature of the workpiece is T_i (°C), and the furnace temperature is T_∞ (°C).

Two questions can be asked: What will be the center temperature after a given time, t_{given}? How long will it take for the center temperature to reach a given temperature, T_{given}? The two questions are related to each other. The first question is to determine the heating/cooling rates, and the second question is to determine the equilibration times. Once the center temperature reaches the furnace temperature, T_∞, then the system can be considered to be at a uniform temperature. It should be mentioned that the center of a solid body can be defined easily for a simple geometry, but can be very difficult to define for a complex geometry. Thus, the problem of predicting equilibration times can be divided into two subproblems, namely, how to define the center of the geometry and how to predict the center temperature.

In many real situations, the volume and heat transfer area of a workpiece can also be difficult to measure, but there are many ways of overcoming this difficulty. Without going further with some extensive listing, one simple method can be recommended for some very complex geometries. For volume measurement, if possible, a solid can be immersed into water, and the water volume of the solid displaced measured. For area measurement, paper can be cut and attached onto the surface, weighing the paper after collecting it from the solid. The weight of the paper will give a satisfactory estimate of the surface area, provided a standard size sheet of paper is weighed.

The overall heat transfer coefficients include the effects of both convective heat transfer and radiative heat transfer. Accurate prediction of the heat transfer coefficient is a difficult problem. Convective heat transfer of the workpiece surface is mainly the result of the turbulent natural convection in most situations. For simple geometries, determination of the average convective heat transfer coefficient can be done by using available empirical relations found in many heat transfer textbooks,[1] but complex surface shapes are not discussed. Reynoldson[3] provided a list of overall heat transfer coefficients for different types of furnaces, which is reproduced in Table 1. These values provide a good starting point for most practical metal heat treatment situations.

It should be noted that the surface area in the input data is only for those subareas where convective or radiative heat exchange occurs with the surrounding air or surfaces. Figure 1 illustrates three ideal situations for a rectangular block with a corner cavity, a surface cavity, and an interior cavity. An external flow field is created over the block. For the corner cavity, the three surfaces are exposed fully to the external air flow and contribute to the surface heat transfer. Thus, these areas should be included in the heat transfer area of the block workpiece. For the interior cavity, the four surfaces do not contribute to the heat transfer between the block and its surroundings. Their areas should not be included in the heat transfer area calculation. In the surface cavity, a flow recirculation can be created. If the flow recirculation is strong, then it will contribute significantly to the surface convective heat transfer, and the cavity surfaces should be included in the heat transfer area. If the cavity is relatively deep and small, then the flow recirculation in the cavity may be very weak (dead zone), and its surfaces will not contribute to the total heat transfer area calculation. At the same time, such a deep cavity may behave like a black hole.

In summary, there are nine input data for the present approach:

- the volume and heat transfer surface area of the workpiece;
- the overall heat transfer coefficient;
- the three physical properties of the workpiece, including the density, the thermal conductivity, and the specific heat capacity; and
- the initial workpiece temperature, the surrounding air temperature, and either the target temperature or the given time.

2.2 Prediction of Equilibrium Times

The new method for prediction of the required time for the center temperature to reach a given temperature can be summarized as follows.

- *Step 1:* Measure the volume and surface area of the workpiece, V and A; calculate the radius of the virtual sphere,

Table 1 Overall heat transfer coefficients for different types of furnaces, from Ref 3

Furnaces	$W/m^2 \cdot °C$
Lead	1200–1800
Salt baths	500–1200
Fluidized beds	500–700
Radiant fluidized beds (above 800°C)	200–300
Forced circulation furnaces	150–200
Radiant atmosphere furnaces (above 750°C)	120–220
Vacuum furnaces	120–200

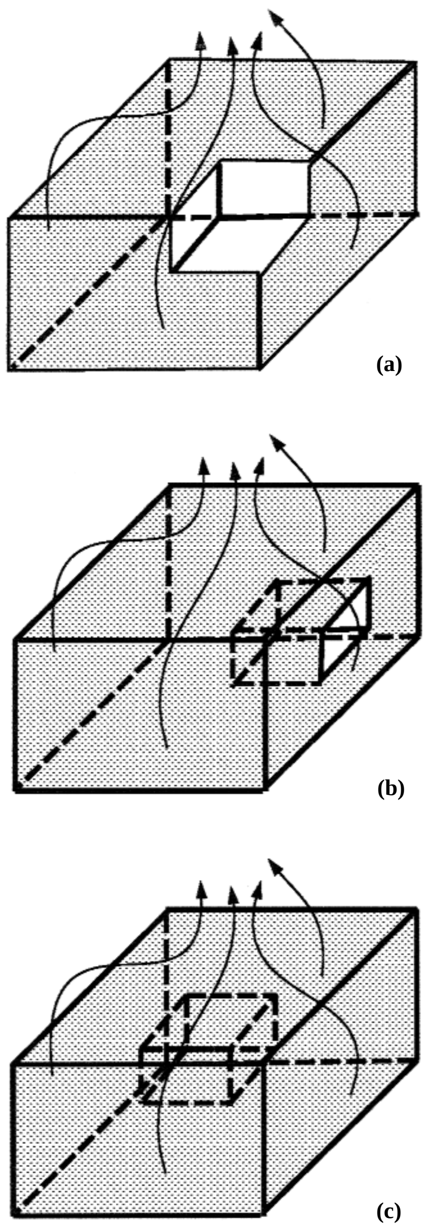

Fig. 1 Three blocks with **(a)** a corner cavity, **(b)** a surface cavity, and **(c)** an interior cavity

R_v; and calculate the thermal diffusivity, α, of the workpiece material:

$$R_v = 3V / A \quad \text{(Eq 1)}$$

$$\alpha = \frac{k}{\rho c} \quad \text{(Eq 2)}$$

- *Step 2:* Calculate the dimensionless Biot number for the virtual sphere:

$$Bi = \frac{hR_v}{k} \quad \text{(Eq 3)}$$

- *Step 3:* Calculate the dimensionless thermal resistance parameter, *C*, by using equation Eq 4 or 5:

$$1 / C = a_1 + b_1 Bi + c_1 Bi^2 + d_1 Bi^3 + e_1 Bi^4 + f_1 Bi^5 \quad \text{(Eq 4)}$$

$$1 / C = a_2 + b_2 Bi \quad \text{(Eq 5)}$$

where a_1 = 0.33309782, b_1 = 0.078429524, c_1 = 0.0051461563, d_1 = –0.00048676155, e_1 = 2.3430387*e* – 05, f_1 = –4.4135362*e* – 07, a_2 = 0.31934443, and b_2 = 0.099154717. Equation 5 is obviously simpler than Eq 4, but offers less accuracy.

- *Step 4:* Calculate the dimensionless temperature, θ, of the workpiece at time, *t:*

$$\theta = \frac{T(t) - T_\infty}{T_i - T_\infty} \quad \text{(Eq 6)}$$

- *Step 5:* Calculate the dimensionless Fourier number, *Fo*, and obtain the required time *t* to reach the final temperature:

$$Fo = \frac{-\ln(\theta)}{C\mathrm{Bi}}$$
$$t = \frac{Fo}{\alpha} R_\upsilon^2$$

2.3 Prediction of Heating/Cooling Rates

The new method for prediction of the center temperature after a given time *t* can be summarized as follows.

- *Step 1:* Measure the volume and surface area of the workpiece, *V* and *A;* calculate the radius of the virtual sphere, R_v; and calculate the thermal diffusivity, α*:*

$$R_v = 3V / A \quad \text{(Eq 7)}$$

$$\alpha = \frac{k}{\rho c} \quad \text{(Eq 8)}$$

- *Step 2:* Calculate the Biot number, *Bi:*

$$Bi = \frac{hR_\upsilon}{k} \quad \text{(Eq 9)}$$

- *Step 3:* Calculate the thermal resistance parameter, *C*, by using Eq 4 or 5.
- *Step 4:* Calculate the Fourier number, *Fo:*

$$Fo = \frac{\alpha t}{R_\upsilon^2} \quad \text{(Eq 10)}$$

- *Step 5:* Calculate the dimensionless temperature, θ, and the temperature, *T*(*t*):

$$\ln(\theta) = -(C\mathrm{Bi})\mathrm{Fo}$$
$$T(t) = (T_i - T_\infty)\theta + T_\infty$$

Note that steps 1 to 3 are the same as those in the prediction of equilibration times.

2.4 Advantages and Disadvantages of the New Methods

The virtual sphere method allows the transient center temperature of any complex geometry to be predicted for a given time or the required time to reach a target temperature to be predicted. The approximate solution for the center temperature of a sphere further simplifies the virtual sphere method for engineering purposes. For clarity, the following notations of different virtual sphere methods based on the approach of predicting sphere center temperatures are introduced. Method 1 uses the approximate solution of a sphere (Eq 4). Method 2 uses the simple approximate solution of a sphere (Eq 5).

The conventional methods, that is, numerical methods, analytical methods, Heisler charts, and rules of thumb, were discussed briefly in the "Introduction." Compared to these methods, the new virtual sphere method has the following advantages.

- The input data are kept to a minimum. This ensures the simplicity of the method.
- The method applies to workpieces of arbitrary geometry, like the numerical methods.
- The method is simple and cost effective to use. It can be easily implemented in a computer program in just a few statements. In fact, hand calculations are also possible with this method, in particular, with method 2.
- The method offers much better accuracy than rules of thumb.

Just like any other methods, the virtual sphere method also has disadvantages, as follows.

- The method is not capable of taking into account the temperature dependence of physical properties.
- The heat transfer coefficient is assumed to be constant.
- The method can only predict the center temperatures, not surface temperatures.
- When variable physical properties or variable heat transfer coefficients need to be considered, numerical methods can be used.

3. Examination of the Virtual Sphere Concept

The key ideas in the new proposed methods are as follows: (a) a virtual sphere is assumed to represent the original complex geometry, and the radius of the virtual sphere, R_v, is taken as $3V/A$; (b) the center temperature of the virtual sphere is assumed to be the center temperature of the real solid body; and (c) the center temperature of a sphere is calculated by a new approximate solution. These ideas are explained and examined in the following section.

3.1 Lumped Heat Capacity Systems

The virtual sphere concept was developed for distributed heat-capacity systems. To understand the virtual sphere concept, the concept of the lumped heat-capacity system will be revisited. If the heat transferred to a body is assumed to be instantaneously and uniformly distributed throughout the body, a lumped heat-capacity system is obtained. For lumped heat-capacity systems, the temperature is uniform within the body, and the solid temperature, $T(t)$, at any time, t, can be obtained analytically as

$$\theta = \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-3\,Bi \times Fo} \qquad \text{(Eq 11)}$$

where T_i is the initial temperature (°C). The *Bi* and *Fo* are two dimensionless numbers: the Biot number, hl/k, and the Fourier number, $\alpha t/l^2$, where $l = 3V/A$ is a characteristic length and $\alpha = k/(\rho c)$.

It should be mentioned that the present choice of the characteristic length is slightly different from the conventional choice, V/A, for a lumped heat-capacity system in some standard text books, for example, Ref 4. The purpose of the current choice is to ensure the radius will be the characteristic length of a solid sphere, which is chosen to be the basic geometry in the present virtual sphere method.

In casting engineering, the volume-to-surface area ratio is often called the modulus.[5] The famous Chvorinov rule shows that the freezing time of any solidifying body depends on its modulus.

Equation 11 shows that the heating or cooling rate in heat treatment is also controlled by the modulus of the solid body. It should be mentioned that the radius of the virtual sphere for a solid body is three times its modulus. The physical meaning of the modulus will be discussed later.

It is commonly accepted that the assumption of a lumped heat-capacity system holds if the Biot number is less than 0.1. Equation 11 shows that ln (θ) is a linear function of the Fourier number if the Biot number is a constant. This is a very important property of lumped heat-capacity problems.

3.2 A Simple Approximate Solution for a Spherical Distributed Heat-Capacity System

When the Biot number is larger than 0.1, the heat transfer problem becomes a distributed heat-capacity system. By studying the three Heisler charts, which apply to distributed heat-capacity systems, it can be concluded for the center temperature in a solid sphere, the temperature at the centerline of an infinitely long cylinder, and the temperature at the midplane of an infinite plate of finite thickness that ln (θ) is also an approximately linear function of the Fourier number for each constant Biot number. The slope of this linear relation is a nonlinear function of the Biot number instead of the linear function of $3Bi$ in the lumped heat-capacity system.

This linear property was used by Gao and Reid[2] to simplify the exact analytical solution of the center temperature in the transient heat conduction of a solid sphere. The approximate solution has a form similar to Eq 11:

$$\theta = \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-C\,\mathrm{Bi} \times \mathrm{Fo}} \qquad \text{(Eq 12)}$$

where C is a thermal resistance parameter. For a lumped heat-capacity system, $C = 3$. For a distributed system, the thermal resistance parameter is a function of the Biot number, as shown by Gao and Reid. Two simple formulas were suggested and both apply to a Biot number range between 0.01 and 20. These were Eq 4 and 5. The $R2$ coefficients of determination are 0.999996 and 0.998483 for Eq 4 and 5, respectively.

When $Bi \leq 0.1$, simply take $C = 3$ as the assumption that a lumped heat-capacity is valid. Equation 5 slightly overpredicts the C value when $\mathrm{Bi} < 1$.

To evaluate the two simple polynomials, 40 different Biot numbers chosen from the Heisler chart were considered with 2,000 Fourier numbers equally spaced from 0.1 to 200 for each Biot number in a solid sphere. Figure 2 shows the predicted

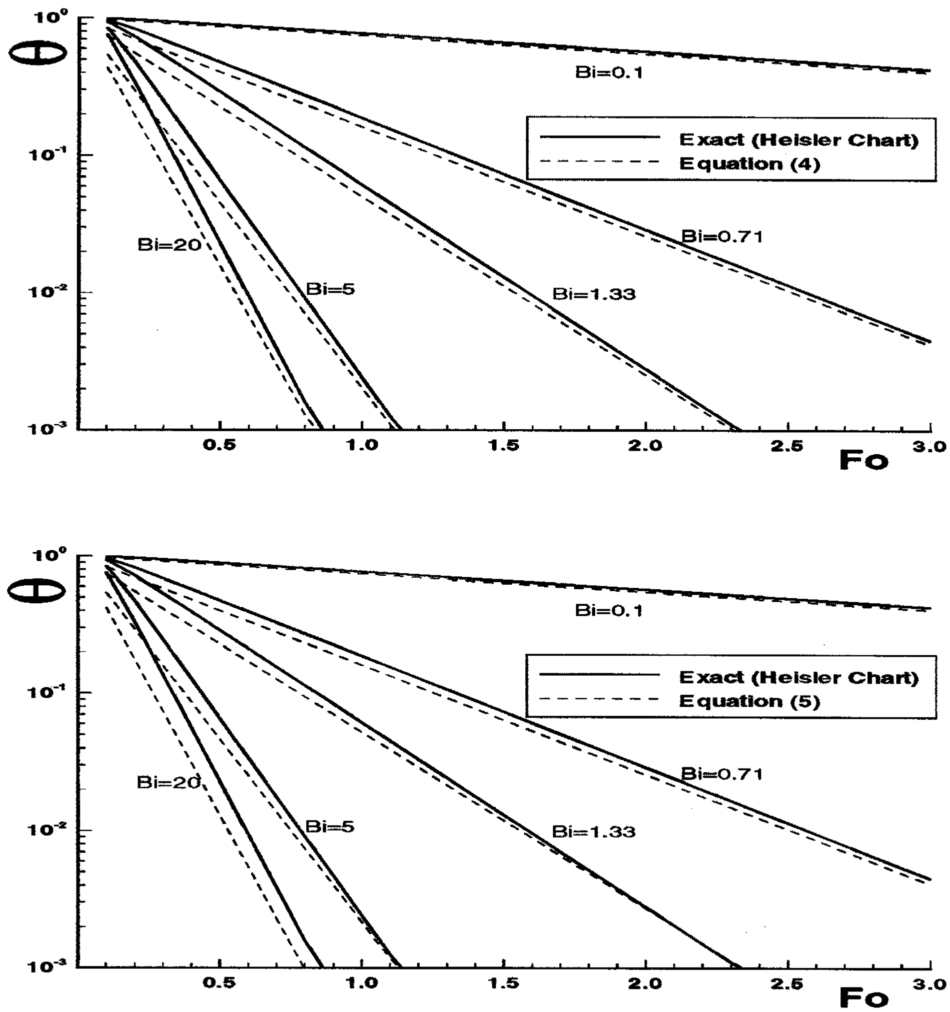

Fig. 2 Dimensionless center temperature in a solid sphere of radius R_v. The predictions by Eq. 4 (top) and 5 (bottom) are compared with the exact solution

Table 2 Summary of errors of predicted dimensionless center temperatures in a solid sphere

Model	Maximum	Average	Root-mean-square
Eq 4	0.322	5.88e-4	4.50e-3
Eq 5	0.338	6.56e-4	4.75e-3

center temperature distributions in a solid sphere by the two different equations of the thermal resistance parameter. Figure 2 is a simple Heisler chart for the sphere center temperature. Figure 2 only plots some representative data for some larger Biot numbers. The accuracy for Biot numbers smaller than 0.1 is much better than those presented in the figures. Table 2 summarizes the maximum, average, and root-mean-square errors for each model. The accuracy shown in Table 2 is considered to be sufficient for engineering heat treatment purposes.

3.3 The Concept of the Virtual Sphere

It should be mentioned that the center of a solid body can be defined easily for simple geometries, but it is difficult to define for complex geometries. The essential idea is that a virtual sphere is assumed to represent the original complex geometry, and the radius of the virtual sphere, R_v, is taken as $3V/A$. Table 3 lists the radius of the virtual sphere for some typical geometries shown in Fig. 3.

The center temperature of the virtual sphere is assumed to be the center temperature of the real solid body. This idea was first formulated by examining the similarity in the Heisler charts for the center temperatures of an infinite plate of finite thickness ($2L$), a solid sphere (radius R), and a long cylinder (radius R). All three geometries are very different from each other, but exhibit similar center temperatures in the Heisler charts. For a constant Biot number, the dimensionless temperature is approximately a linear function of the Fourier number. If a consistent definition of the characteristic length is chosen for each geometry, $3L/2$ for the infinite plate, R for the sphere, and 1.5 R for the

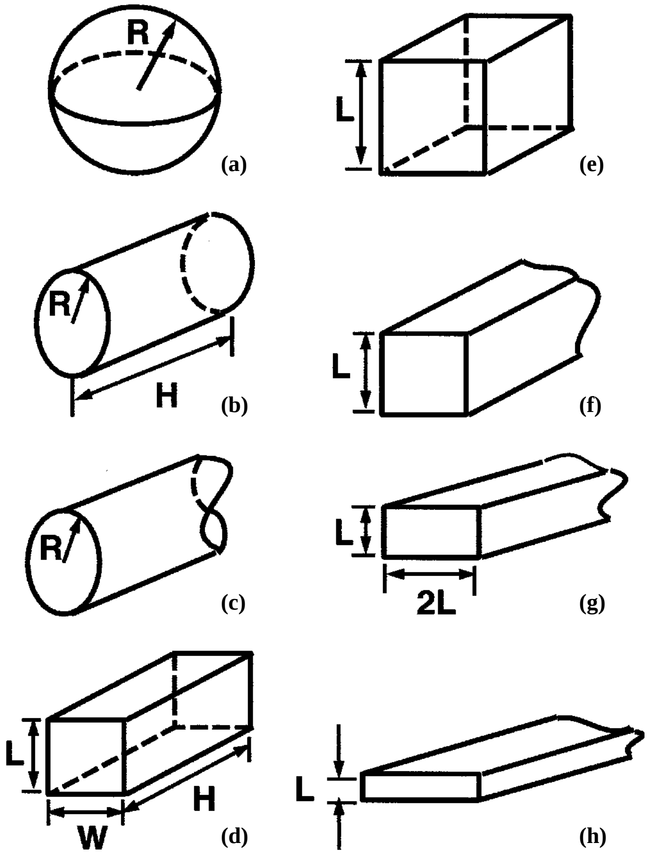

Fig. 3 Some typical geometries: **(a)** sphere, **(b)** short cylinder, **(c)** long cylinder, **(d)** rectangular prism, **(e)** cube, **(f)** long square, **(g)** long rectangle ($L \times 2L$), and **(h)** infinite plate

Table 3 The radius of the virtual sphere for some typical geometries

Geometry	Radius of the virtual sphere
Sphere	R
Short cylinder	$3R(2R/H + 2)^{-1}$
Long cylinder	$1.5R$
Rectangular prism	$(3WHL)/[2(HL + HW + WL)]$
Cube	$0.5L$
Long square	$3L/4$
Long rectangle ($L \times 2L$)	L
Infinite plate	$3L/2$

long cylinder, it can be seen immediately that the three Heisler charts become almost identical in the practical Biot number range between 0 and 20 if the previously given definition of the characteristic length is used. The relative difference in Fourier prediction when $\theta = 0.001$ is 8% at $Bi = 1$. It increases to 45% at $Bi = 10$, 50% at $Bi = 20$, and 60% at infinite Biot number. There is almost no difference for a Biot number less than 0.1, which is a lumped heat-capacity system.

This is a promising similarity because the three geometries are very different from each other. If the sphere is taken as a basic geometry for comparison, the long cylinder and the infinite plate could be considered to be very nonspherical. The similarity in the center temperature histories for these three very different geometries suggests that the center temperature history for one geometry (*e.g.,* long cylinder) can be predicted from the Heisler chart of another geometry (*e.g.,* sphere).

In fact, if the infinite plate is chosen as the basic geometry, then a virtual infinite plate can be assumed to represent the original complex geometry, and the half-thickness of the virtual infinite plate is taken as V/A. This is the physical meaning of the modulus used in casting engineering mentioned earlier. However, because a real sphere provides the minimum surface area, which gives less heat transfer capacity, the equilibration time calculated by the virtual sphere method based on the solution for a sphere is the most conservative value. Thus, the sphere is chosen as the basic geometry in the present method. At the same time, the sphere is a finite geometry, while the infinite plate is somewhat imaginary. This means that if a constant volume of a workpiece is provided, a sphere can be created, but not an infinite plate in practical situations.

This is the basis for the assumption that the center temperature of the virtual sphere is approximately that of an arbitrary solid body for the practical Biot number range between 0 and 20. The physical meaning of the virtual sphere can be easily understood by examining the definition of its radius.

When a spherical workpiece evolves into another geometry of the same volume, its surface area increases and the radius of the virtual sphere decreases. Under the same heat transfer conditions, a reduction in the radius of the virtual sphere means a decrease in equilibration times. This is due to an increase of the heat transfer surface area.

Thus, only the center temperature solution for a solid sphere is needed for determining the equilibration times. This solution is available in an exact form, but requires numerical methods to compute. The Heisler chart for a sphere can also be used. The approximate solutions presented earlier are used in the present method.

4. Evaluation I: Exact Solutions

The following three examples are the modified versions of the example problems from the textbook by Kakac and Yener.[4] They can also be solved by using the Heisler charts. The shapes of the workpieces in these examples are very nonspherical, and thus, they can serve as critical test cases for the new methods.

In example 1, a steel cylinder, 20 cm in diameter and 2 m in length, heated initially to a temperature of 500 °C, was suddenly immersed in an oil bath maintained at 20 °C. For the following conditions, calculate the time required for the center temperature of the steel cylinder to reach 20.5 °C: $\rho = 7{,}700$ kg/m^3, $c = 500$ J/kg · °C, $k = 41$ W/m · °C, and $h = 1{,}200$ W/m^2 · °C.

Table 4 summarizes the solutions with different methods. It should be mentioned that in the Heisler chart method, the radius of the cylinder is used as the characteristic length, and in methods 1 and 2, the radius of the virtual sphere is used as the characteristic length. The largest error is about 21% for a rather high Biot number (>>0.1) with method 2.

Table 4 Summary of the solutions for the three examples using three different methods

	Parameters	Example 1	Example 2	Example 3
Basic data	$\alpha = k/\rho c$ (m^2/s)	1.065e-5	4.746e-6	1.065e-5
	$R_\upsilon = 3V/A$ (m)	0.149	0.9	0.25
	$Bi = hR_\upsilon/k$	4.361	0.581	0.549
	$\theta = (T_{given} - T_\infty)/(T_i - T_\infty)$	0.0001	. . .	0.001
	$Fo - \alpha t/R_\upsilon^2$	. . .	0.570	. . .
Method 1	C from Eq 4	1.351	2.630	2.648
	$Fo = -\ln(\theta)/CB$i	1.173	. . .	4.753
	$t = FoR_\upsilon^2/\alpha$ (s)	**2,445**	. . .	**27,895**
	$\theta = e^{(-CBi} \times$ Fo)	. . .	0.419	. . .
	$T = \theta(T_i - T_\infty) + T_\infty$ (°C)	. . .	**344**	. . .
Method 2	C from Eq 5	1.330	2.630	2.676
	$Fo = \ln(\theta)\ CBi$	1.191	. . .	4.705
	$t = FoR_\upsilon^2/\alpha$ (s)	**2,482**	. . .	**27,611**
	$\theta = e^{(-CBi} \times$ Fo)	. . .	0.416	. . .
	$T = \theta(T_i - T_\infty) + T_\infty$ (°C)	. . .	**346**	. . .
Heisler chart	Bi $= hR_\upsilon/k$	2.93	0.194	0.549
	Fo (obtained from H chart)	2.15	. . .	5.0
	Fo $= \alpha t/L^2$	. . .	5.126	. . .
	t = For$^2_0/\alpha$ (s)	**2,046**	. . .	**29,344**
	θ (obtained from H chart)	. . .	0.4	. . .
	$T = \theta(T_i - T_\infty) + T$ (°C)	. . .	**355**	. . .

The solutions obtained with methods 1 and 2 are compared with those obtained by the Heisler charts (H charts).

In example 2, a 60 cm (2L) thick plane wall at a uniform temperature of 21 °C was suddenly exposed on both sides to a hot gas stream at 577 °C. For the following conditions, calculate the temperature at the midplane after 27 h of heating: $\rho = 2{,}600$ kg/m^3, c = 1,256 J/kg · °C, k = 15.5 W/m · °C, and h = 10 W/m^2 · °C.

Again, Table 4 summarizes the solutions. In calculating the radius of the virtual sphere, only the surface area of each side of the plane wall is considered because the workpiece is very large. In the Heisler chart method, the half-thickness is used as the characteristic length. The largest error (about 3%) occurs in method 1 for a Biot number of about 0.6.

Example 3 was a steel cylinder 50 cm in diameter and 50 cm long, initially heated at a uniform temperature of 24 °C, and then placed in a furnace maintained at 930 °C. Estimate the time required for the center temperature to reach 929°C under the following conditions: ρ = 7,700 kg/m^3, c = 500 J/kg · °C, k = 41 W/m · °C, and h = 90 W/m^2 · °C. Table 4 lists the solutions. This example differs from the other two examples because there is no Heisler chart available for a short cylinder. Here, the two-dimensional temperature distribution in the cylinder is obtained by the Heisler charts of an infinite plate and cylinder. In the infinite plate and cylinder cases, characteristic lengths are the radius and the half-thickness, respectively, which happens to be the same for this problem, that is, 0.25 m, which means that the resulting Fourier numbers should also be the same.

It can be shown that the $\theta_{\text{short cylinder}} = \theta_{\text{long cylinder}} \times \theta_{\text{infinite plate}}$.[4] By trial and error, from the Heisler charts, it was found that if Fo = 5.0, $\theta_{\text{long cylinder}} = 0.01$ and $\theta_{\text{infinite plate}} = 0.1$, which satisfy the value of θ for the present problem. From the Fourier number, it was finally found the required time is 29,344 s, which agrees well with the predicted results by the proposed virtual sphere concepts.

It should be mentioned that the percentage deviation used in the previous comparisons was for general purposes only. Different Biot numbers were tested. A small error in example 2 compared to those in examples 1 and 3 is mainly due to a smaller Biot number in example 2. Based on these results, conclusions should not be simply drawn that the method is not very suitable for long cylinders, but suitable for plates.

The previous three examples show that the virtual sphere approach can be successfully applied to geometries that are far from that of a sphere. The new methods are much simpler than any existing method, except those using rules of thumb, which are not accurate and generally applicable.

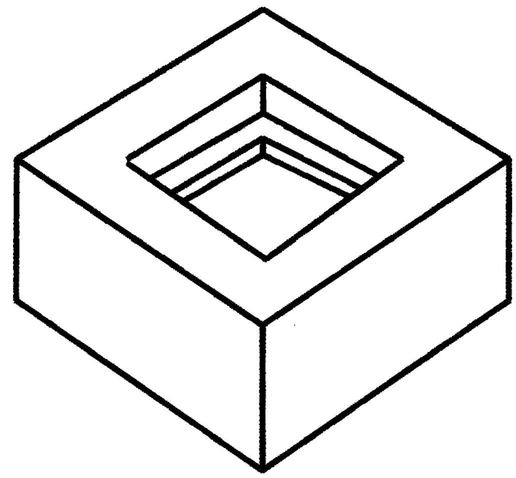

Fig. 4 The H13 die used in the experiment of Killian[5]

5. Evaluation II: Experimental Data

The developed method was further demonstrated *via* a practical example where measured temperature variation profiles were available.

For example 4, when investigating the distortion of H13 dies heated in fluidized bed furnaces, Killian[6] recorded experimentally the temperature variation of two identical H13 steel blocks during both heating and cooling in a heat treatment cycle. The two H13 dies were heated in two furnaces, a fluidized bed and a vacuum furnace with a subatmospheric quench, that is, 700 torr of nitrogen with a 50 horsepower gas recirculation fan. Six thermocouples were placed in various positions on each of the two dies. Figure 4 shows the shape of the die. The block had an overall dimension of 300 by 300 by 150 mm. A cavity was machined at the center of one side of the block. The dimension of the cavity was 160 by 160 by 50 mm. A second smaller cavity was machined at the bottom of the larger cavity, which had a dimension of 75 by 75 by 15 mm.

Figure 5 shows the measured heating and cooling curves for both dies. It should be mentioned that the heat treatment cycles

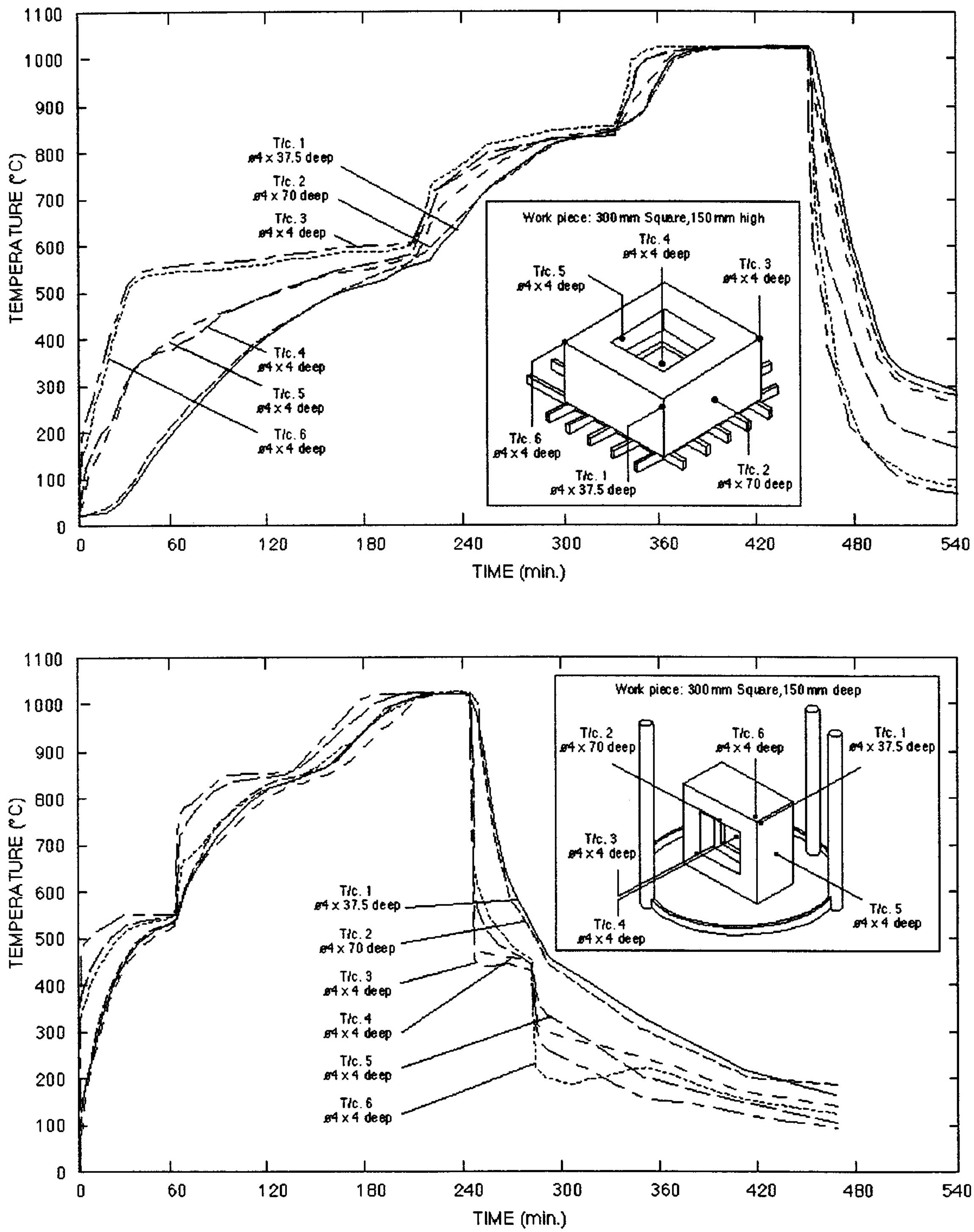

Fig. 5 Measured heating and cooling curves for an H13 die in the vacuum furnace (top) and in the fluidized bed furnace (bottom)

for both dies were slightly different. There was no step quench 1 at 400 °C with the vacuum furnace cycle. The problem was to predict the heating and cooling curves for both dies.

For the solution of this realistic case, the practical problems of temperature-dependent thermal properties and heat transfer coefficients were faced. The following constant physical properties were assumed: $\rho = 7{,}724$ kg/m^3, $c = 622$ J/kg · °C, and $k = 28.5$ W/m · °C.

With regard to heat transfer coefficients, it is obvious that for different stages of heat treatment, they are different. Without detailed information, a constant heat transfer coefficient was assumed, which is obtained from Table 1. For the fluidized bed furnace, the value was taken as 500 W/m^2 · °C, and for the vacuum furnace, 120 W/m^2 · °C. To demonstrate the sensitivity of the heat transfer coefficients, another set of heat transfer coefficients was also considered, 350 W/m^2 · °C for the fluidized bed furnace and 200 W/m^2 · °C for the vacuum furnace.

In simulating the heat treatment cycles, the center temperature of the virtual sphere was calculated at different times. At the end of each stage (that is, first preheat, second preheat, harden temperature, and step quench 1), the initial temperature of the new stage was simply taken as the final center temperature of the previous stage. This is true if the previous stage reaches equilibrium status.

The diameter of the virtual sphere was calculated as 0.092 m, and the Biot number considered ranged from 0.37 to 1.54. These Biot numbers indicated that the system cannot be considered as a lumped heat-capacity system. Figure 6 shows the predicted heating and cooling curves. In these predictions, the total time interval of 540 min was divided into 2,000 intervals.

It can be seen that the differences between heating and cooling rates in the two furnaces were well predicted. Although it is hard to find a corresponding location in the real die that has the same temperature as the center of the virtual sphere, it seems that locations of thermocouples T/c.1 and T/c.2, which are deep in

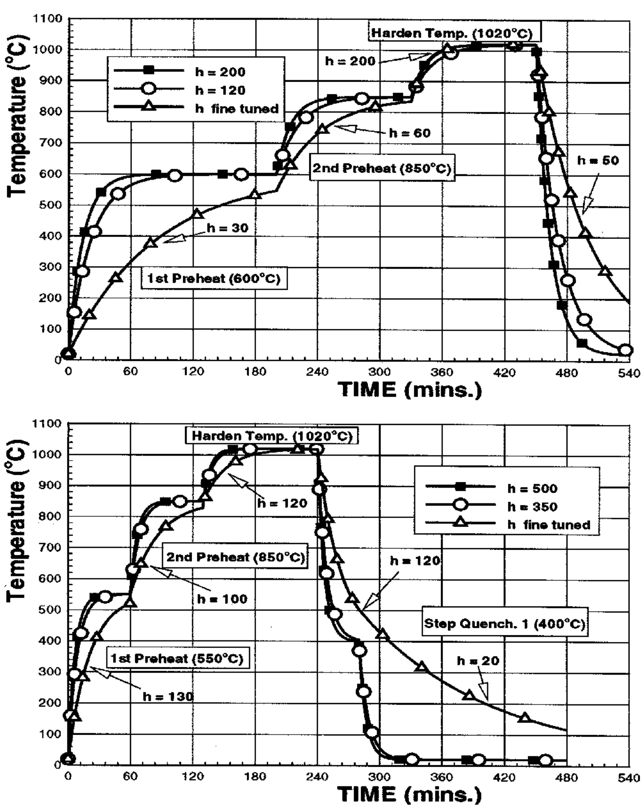

Fig. 6 Predicted heating and cooling curves in the vacuum furnace (top) and in the fluidized bed furnace (bottom). In each furnace, two different heat transfer coefficients are used as listed in the table in each figure

the samples, may be a good assumption. The present method cannot predict the surface temperatures.

For the first and second preheating stages, the predicted curves indicate that the dies reach their equilibrium while the measured curves indicate the opposite. This shows that the heat transfer coefficients used in the prediction are too high for these two stages. The same conclusions also apply to the cooling curves. The results indicate that an accurate characterization of the heating and cooling processes in a furnace is very important. However, it is quite often very difficult. Accurate engineering prediction models are required to determine the overall heat transfer coefficients, based on furnace operating parameters. The overall heat transfer coefficients are necessary inputs for the present method and any other numerical methods.

Heat transfer coefficients can vary at different stages of heating and cooling, for example, as in the experiments of Killian.[6] Figure 6 shows a fine tuned prediction. Some very unrealistic *h* values were used. It was noticed that in the vacuum furnace case, the experiment showed that there was a very high gradient in the workpiece, indicating a high heat conduction rate in the workpiece and a high convective heat transfer rate at surfaces. However, a very small heat transfer coefficient was suggested by a fine tuning simulation in Fig. 6. This may suggest that some characteristic of the furnace may not have been documented by Killian and, thus, may not have been considered here.

6. Conclusions

A practical engineering method has been presented and evaluated in this paper to estimate the equilibration times and the heating/cooling rates for heat treatment. The two most important components in this new approach are the virtual sphere concept of representing any geometry and two new simple approximate solutions for the center temperature of a solid sphere. The methods developed in this paper were evaluated against the exact solutions of some regular geometries, which are far from that of a sphere and an experimentally measured temperature profile of realistic dies in a heat treatment cycle. The analytical tests show that the largest error in estimating equilibration times in a range of Biot numbers between 0.1 and 5 is about 20% for very nonspherical geometries. A fairly good agreement is obtained between the results predicted by the new virtual sphere method and the measurement. The new method is capable of treating any arbitrary geometry in a rather simple manner. Before the virtual sphere method is recommended for engineering applications, reliable engineering prediction methods for determining the heat transfer coefficients in different furnaces are needed.

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